Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf Problem sheet 0

Problem 0 (4 points). Two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ are said to be *forcing equivalent*, if every $\mathbb{P}$-generic extension of M is also a $\mathbb{Q}$-generic extension of M, and vice versa every $\mathbb{Q}$-generic extension of M is also a $\mathbb{P}$-generic extension of M.
Let $\mathbb{P}$ be the forcing notion whose conditions are intervals $[p, q]$ of the real line $\mathbb{R}$ with rational endpoints $p < q$, partially ordered by inclusion.

(a) Let G be M-generic for $\mathbb{P}$. Prove that G can be reconstructed from a single real.

(b) Show that $\mathbb{P}$ is forcing equivalent to Cohen forcing $\mathbb{C}$.

Problem 1 (8 Points). *Random forcing* $\mathbb{P}$ is defined as the set of Borel subsets p of the real line $\mathbb{R}$ with positive Lebesgue measure $\mu(p) > 0$, ordered by inclusion.

(a) Show that $\langle \mathbb{P}, \leq \rangle$ satisfies the c.c.c.

(b) Let G be M-generic for $\mathbb{P}$. Find a canonical real (denoted *Random real*) which is in $M[G]$ but not in M.

(c) Let $X = \bigsqcup_{n \in \omega} X_n$ be the disjoint union of Lebesgue measurable sets X_n and $\mu(X) < \infty$. Show that for every $\varepsilon > 0$ there is $n_0 \in \omega$ with $\mu(X) - \mu(\bigsqcup_{n<n_0} X_n) < \varepsilon$.

(d) Given $\varepsilon > 0$, a $\mathbb{P}$-name σ and $p \in \mathbb{P}$ with $\mu(p) < \infty$ and $p \Vdash_{\mathbb{P}}^M \sigma \in \check{\omega}$, prove that there is $n \in \omega$ and $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}}^M \sigma \leq \check{n}$ and $\mu(p \setminus q) < \varepsilon$.

(e) Suppose that $\varepsilon > 0$ and $p \Vdash_{\mathbb{P}}^M \dot{f} : \omega \to \omega$ with $\mu(p) < \infty$. Prove that there are $q \leq p$ in $\mathbb{P}$ and a ground model function $g : \omega \to \omega$ such that $q \Vdash_{\mathbb{P}}^M \forall n \in \omega (\dot{f}(n) \leq g(n))$ and $\mu(p \setminus q) < \varepsilon$.

(f) Let $f \leq^* g \iff \exists n_0 \in \omega \forall n \geq n_0 (f(n) \leq g(n))$ for $f, g \in {}^\omega\omega$. Show that $\mathbb{P}$ is ω^ω*-bounding* over M, i.e. if G is M-generic for $\mathbb{P}$ and $f \in ({}^\omega\omega)^{M[G]}$, then there is some $g \in ({}^\omega\omega)^M$ with $f \leq^* g$.

Problem 2 (4 points). Let $\mathbb{C}$ denote Cohen forcing and let G be M-generic for $\mathbb{C}$. Consider the Cohen real $c = \bigcup G : \omega \to 2$ and let $c_0, c_1 : \omega \to 2$ be given by $c_0(n) = c(2n)$ and $c_1(n) = c(2n+1)$ for all $n \in \omega$.

(a) Prove that $\langle c_0, c_1 \rangle$ is M-generic for the product $\mathbb{C} \times \mathbb{C}$, i.e. c_0 and c_1 are also Cohen reals.

(b) Use (a) to show c_1 is not in $M[c_0]$.

Problem 3 (4 points)**.** Let $\mathbb{P}$ be a forcing which satisfies the c.c.c. Show that stationary subset of ω_1 in M remain stationary in $\mathbb{P}$-generic extensions $M[G]$.

Hint: If $\dot{C}$ is a name for a club subset of ω_1 and $p \Vdash^M_{\mathbb{P}}$ "$\dot{C}$ is club" then prove that $\bar{C} = \{\alpha < \omega_1 \mid p \Vdash^M_{\mathbb{P}} \alpha \in \dot{C}\} \in M$ is a club subset of ω_1.

Please hand in your solutions on Monday, 02.11.2015 before the lecture.